

Laplace equation in spherical coordinates I + II

Laplace equation is given by

$$\nabla^2 V = 0$$

In spherical coordinates (r, θ, ϕ) , Laplace equation is written as

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r^2 V) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0 \quad \text{--- (1)}$$

We use separation of variables to find the solution of above equation. We assume that potential is written in the product form as

$$V(r, \theta, \phi) = \frac{R(r)}{r} P(\theta) Q(\phi) \quad \text{--- (2)}$$

To simplify the algebra we have included an extra factor $\frac{1}{r}$. Next, using eq. (2) in equation (1)

$$\begin{aligned} \frac{1}{r} \frac{\partial^2}{\partial r^2} \left[r \cdot \frac{R(r)}{r} P(\theta) Q(\phi) \right] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial}{\partial \theta} \left(\frac{R(r)}{r} P(\theta) Q(\phi) \right) \right] \\ + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \left[\frac{R(r)}{r} P(\theta) Q(\phi) \right] = 0 \end{aligned}$$

$$\text{or } P(\theta) Q(\phi) \frac{1}{r} \frac{\partial^2 R(r)}{\partial r^2} + \frac{R(r)}{r} Q(\phi) \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial P(\theta)}{\partial \theta} \right) \\ + \frac{R(r)}{r} P(\theta) \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 Q(\phi)}{\partial \phi^2} = 0$$

Next, we multiply above expression by $\frac{r^3 \sin \theta}{R(r)P(\theta)Q(\phi)}$

we obtain,

$$\frac{r^2 \sin^2 \theta}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{\sin \theta}{P(\theta)} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) + \frac{1}{Q(\phi)} \frac{d^2 Q(\phi)}{d\phi^2} = 0 \quad (3)$$

The third term is independent of r and θ , thus it ~~is~~ must be equal to a constant.

$$\frac{r^2 \sin^2 \theta}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{\sin \theta}{P(\theta)} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) - m^2 = 0 \quad (4)$$

and $\frac{1}{Q(\phi)} \frac{d^2 Q(\phi)}{d\phi^2} = -m^2 \quad (5)$

Eq. (5) can be solved.

$$\frac{d^2 Q(\phi)}{d\phi^2} = -m^2 Q(\phi)$$

$$\left(\frac{d^2}{d\phi^2} + m^2 \right) Q(\phi) = 0$$

Solution is $Q(\phi) = A_m e^{im\phi} + B_m e^{-im\phi}, m \neq 0$

for $m=0$, soln $\rightarrow Q(\phi) = A_0 + B_0 \phi$

Again taking Eq. (4)

$$\frac{r^2 \sin^2 \theta}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{\sin \theta}{P(\theta)} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) - m^2 = 0$$

dividing by $\sin^2 \theta$,

$$\frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{1}{\sin^2 \theta P(\theta)} \frac{d}{d\theta} \left(\sin^2 \theta \frac{dP(\theta)}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = 0 \quad (7)$$

The last term and first term are independent and therefore we can write

$$\frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} = l(l+1) = 0 \quad (8)$$

$$\text{where } \frac{1}{\sin^2 \theta P(\theta)} \frac{d}{d\theta} \left(\sin^2 \theta \frac{dP(\theta)}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = -l(l+1) \quad (9)$$

Now eqs (8) & (9) can be written as

$$\frac{d^2 R(r)}{dr^2} = l(l+1) \frac{R(r)}{r^2}$$

$$\text{and } \frac{d}{d\theta} \left[\sin^2 \theta \frac{dP(\theta)}{d\theta} \right] + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] \sin^2 \theta P(\theta) = 0$$

For the last equation we take $x = \cos \theta$, which

gives $\frac{d}{d\theta} = -\sqrt{1-x^2} \frac{d}{dx}$. Therefore, we write

$$\frac{d^2 R(r)}{dr^2} = l(l+1) \frac{R(r)}{r^2} \quad (10)$$

$$\text{and } \frac{d}{dx} \left[(1-x^2) \frac{dP(x)}{dx} \right] + \left[l(l+1) - \frac{m^2}{1-x^2} \right] P(x) = 0 \quad (11)$$

Let us try solution of the form $R(r) = r^\alpha$ of eqn. (10)

$$\frac{d^2 (r^\alpha)}{dr^2} = l(l+1) \frac{r^\alpha}{r^2}$$

$$\alpha(\alpha-1)r^{\alpha-2} = l(l+1)r^{\alpha-2}$$

$$\text{or } [\alpha^2 - \alpha - l(l+1)] r^{\alpha-2} = 0$$

$$\text{or } \alpha^2 - \alpha - l(l+1) = 0$$

which gives $\alpha = l+1$, $\alpha = -l$

Therefore the general solution is given by

$$R(r) = A_l r^{l+1} + B_l r^{-l}$$

Now we write the general solution of eqn. (1) (see eqn. (2)).

$$V(r, \theta, \phi) = \sum_m \sum_l (A_l r^{l+1} + B_l r^{-l-1}) (A_m e^{im\phi} + B_m e^{-im\phi}) P_l^m(\cos\theta)$$

where $P_l^m(\cos\theta)$ is the solution of the equation having θ dependence, the equation (11). (12)

For the special case $m=0$, we solve equation (11) for $P(x)$.

The eqn. (11) for $m=0$ is

$$\frac{d}{dx} \left[(1-x^2) \frac{dP(x)}{dx} \right] + l(l+1) P(x) = 0 \quad \text{--- (13)}$$

Let us try a power series solution of the form

$$P(x) = \sum_{j=0}^{\infty} a_j x^{j+\alpha} \quad \text{--- (14)}$$

Using (14) in (13),

$$\frac{d}{dx} \left[(1-x^2) \frac{d}{dx} \sum_j a_j x^{j+\alpha} \right] + l(l+1) \sum_j a_j x^{j+\alpha} = 0$$

$$\frac{d}{dx} \left[(1-x^2) \sum_j a_j (j+\alpha) x^{j+\alpha-1} \right] + l(l+1) \sum_j a_j x^{j+\alpha} = 0$$

$$\sum_j (j+\alpha)(j+\alpha-1) a_j x^{j+\alpha-2} - \sum_j (j+\alpha)(j+\alpha+1) a_j x^{j+\alpha} + l(l+1) \sum_j a_j x^{j+\alpha} = 0$$

$$\sum_{j=0}^{\infty} (j+\alpha)(j+\alpha-1) a_j x^{j+\alpha-2} - \sum_{j=0}^{\infty} [(j+\alpha)(j+\alpha+1) - l(l+1)] a_j x^{j+\alpha} = 0$$

Combining the same powers in one sum

$$\alpha(\alpha-1) a_0 x^{\alpha-2} + (1+\alpha)\alpha a_1 x^{\alpha-1} + \sum_{j=2}^{\infty} [(j+\alpha+2)(j+\alpha+1) a_{j+2} - \{(j+\alpha)(j+\alpha+1) - l(l+1)\} a_j] x^{j+\alpha} = 0$$

above expressions is true for all x , therefore, coefficients of each power is equal to zero

$$\alpha(\alpha+1) a_0 = 0 \quad \text{--- (15)}$$

$$(1+\alpha)\alpha a_1 = 0 \quad \text{--- (16)}$$

$$(j+\alpha+2)(j+\alpha+1) a_{j+2} - [(j+\alpha)(j+\alpha+1) - l(l+1)] a_j = 0$$

↓

$$a_{j+2} = \frac{(j+\alpha)(j+\alpha+1) - l(l+1)}{(j+\alpha+2)(j+\alpha+1)} a_j \quad \text{--- (17)}$$

Eq. (17) is the recurrence relation. From equations (15) & (16) $a_0 = 0$

from Eq. (12) $\rightarrow a_{\text{odd}} = 0$. Thus.

$$P_e(x) = \sum_{j=0, \text{even}}^{\infty} a_j x^{j+\alpha} \quad \text{--- (18)}$$

To satisfy the eq. (17) we have two choices of α .

$$\alpha = 0 \text{ or } \alpha = 1$$

Solution converges only for $x=1$ if series is finite and series will only have finite number of terms if coefficients in eq. (17) are zero at some point

$$\frac{(J_{\text{max}} + \alpha)(J_{\text{max}} + \alpha + 1) - l(l+1)}{(J_{\text{max}} + \alpha + 2)(J_{\text{max}} + \alpha + 1)} = 0$$

Now for $\alpha = 0$,

$$J_{\text{max}}(J_{\text{max}} + 1) - l(l+1) = 0$$

$$\Rightarrow J_{\text{max}} = l; \text{ which gives}$$

$$P_e(x) = \sum_{j=0, \text{even}}^l a_j x^j$$

or

$$P_e(x) = \sum_{j=0}^{l/2} a_{2j} x^{2j}$$

$$\text{with } a_{j+2} = \frac{j(j+1) - l(l+1)}{(j+2)(j+1)} a_j, \text{ for } l = \text{even}$$

(iv) For $\alpha = 1$.

$$(J_{\text{max}} + 1)(J_{\text{max}} + 2) - l(l+1) = 0$$

which gives $J_{\text{max}} = l - 1$

$P_l(x)$ in this case is given by

$$P_l(x) = \sum_{j=0, \text{even}}^{l-1} a_j x^{j+1}$$

or

$$P_l(x) = \sum_{j=0}^{(l-1)/2} a_{2j} x^{2j+1}$$

with $a_{j+2} = \frac{(j+1)(j+2) - l(l+1)}{(j+3)(j+2)} a_j$ for l odd

(2)

Writing the explicit form of $P_l(x)$

$$P_0(x) = a_0, \quad P_1(x) = a_0 x$$

$$P_2(x) = a_0 + a_2 x^2 \quad \text{with } a_2 = -3a_0$$

$$P_3(x) = a_0 x + a_2 x^3 \quad \text{with } a_2 = -\frac{5}{3} a_0$$

or $P_0(x) = a_0, \quad P_1(x) = a_0 x$

$$P_2(x) = a_0 (1 - 3x^2)$$

$$P_3(x) = a_0 \left(x - \frac{5}{3} x^3\right)$$

a_0 is arbitrary, conventionally we choose it such that $P_l(1) = 1$. Thus,

Now $P_0(x) = 1,$

$$P_1(x) = x,$$

$$P_2(x) = \frac{1}{2} (3x^2 - 1)$$

$$P_3(x) = \frac{1}{2} (5x^3 - 3x)$$

$P_l(x)$ is defined by Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l \quad \text{--- (22)}$$

It follows the recurrence relations.

$$P_{l+1}(x) = \frac{2l+1}{l+1} P_l(x) - \frac{l}{l+1} P_{l-1}(x); \text{ and}$$

$$P_l(x) = \frac{l}{2l+1} \frac{d}{dx} [P_{l+1}(x) - P_{l-1}(x)]$$

The orthogonality condition is given by

$$\int_{-1}^1 P_l(x) P_l(x) dx = \frac{2}{2l+1} \delta_{kl}$$

Finally, we write the general solution of Laplace equation in spherical coordinates for the case $m=0$.

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-l-1}) P_l(\cos \theta) \quad \text{--- (23)}$$

where $P_l(x)$ are the Legendre Polynomials defined by eq. (22).